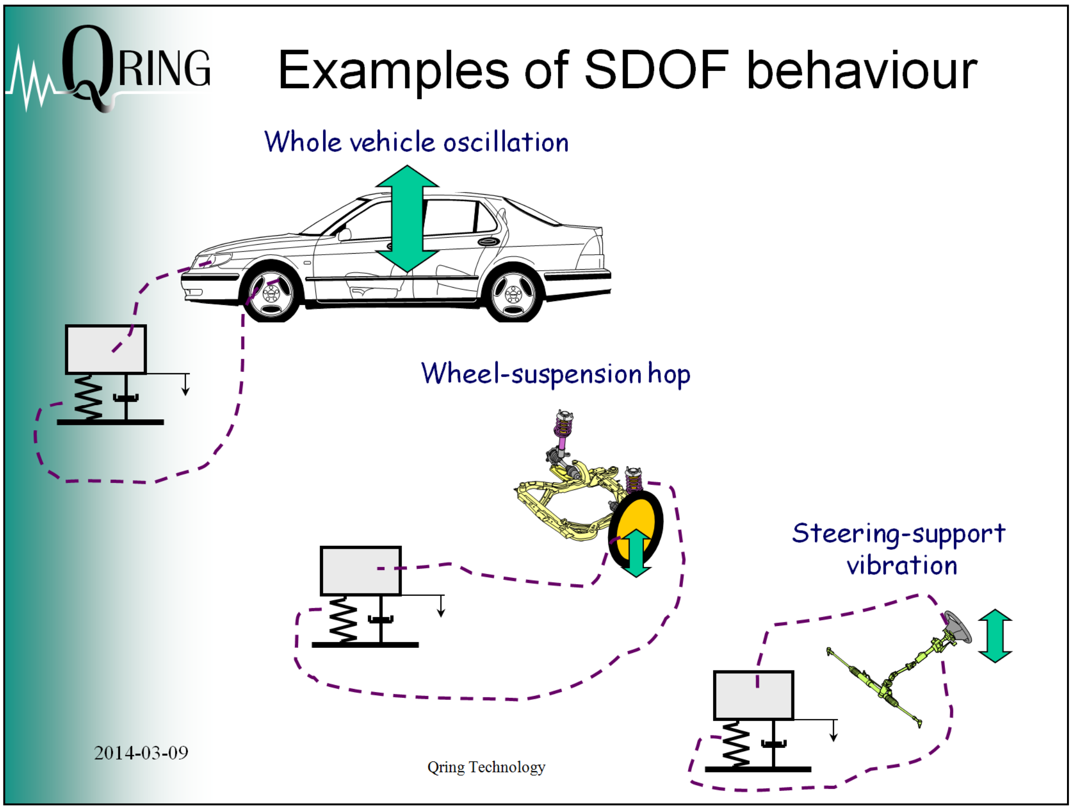
Structural dynamics for Single Degree of Freedom (SDOF) systems

This chapter introduces you to the most useful mechanical oscillator model, a mass-spring system with a single degree of freedom. Basic understanding of this system is the gateway to the understanding of oscillation of more complex systems.

The reason for the importance of the simple mass-spring system is because it is convenient to express the oscillation of complicated cases a the sum of several mass-spring systems. This will be discussed in the multi-degrees of freedom chapter.



The top example shows whole vehicle oscillation where the mechanical mass is the the vehicle weight minus the wheels. The mechanical spring stiffness is the vehicle suspension.

The mid example is a local wheel-suspension resonance where the weight of the wheel (brakes etc.) is the mechanical mass and the stiffness is the vehicle suspension.

The example at the bottom is the steering-support resonance where the inertia is that of the steering wheel and steering column. The stiffness is the the car body stiffness.

Some mathematics

- A harmonic displacement, x that varies with the frequency, ω and has the amplitude, $\hat{x}$ can be written as

$$x = \hat{x}e^{j\omega t}$$

- The time derivative of the displacement gives us the velocity and acceleration

$$x = \hat{x}e^{j\omega t} \Rightarrow u = \frac{dx}{dt} = j\omega\hat{x}e^{j\omega t} \Rightarrow a = \frac{du}{dt} = -\omega^2\hat{x}e^{j\omega t}$$

- Therefore,

$$u = j\omega x \quad \text{and} \quad a = -\omega^2 x$$

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We need to introduce complex notation to work with vibration. The expression

$$e^{j\omega t} = cos(\omega t) + jsin(\omega t)\ ,$$

which tells us that the complex notation allows us to work with sine and cosine oscillation simultaneously. Complex notation shortens expressions that involve oscillation and is therefore convenient to work with. However, you need not concern yourself with complex notation or complex mathematics in this course.

The boxed equations are central for work with vibration. *Try to memorise that this is how we convert between displacement, velocity and acceleration for steady state oscillation.*

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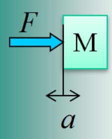
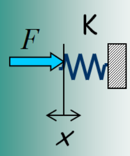
Some mechanics



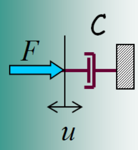
- The reaction when we excite a mass with a sinus force is:

  (time domain) $F = Ma \Rightarrow F = -\omega^2 MX$ *(frequency domain)*



- The reaction when we excite a mass-less spring with a sinus force is:

  (time domain) $F = KX \Rightarrow F = KX$ *(frequency domain)*



- The reaction when we excite a perfect viscous damper with a sinus tone is:

  (time domain) $F = Cu \Rightarrow F = j\omega CX$ *(frequency domain)*

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The top example shows how we can write the force reaction of a mass that oscillates with the frequency f (=> $\omega = 2\pi f$). The equation reveals that we need more force to oscillate the mass with the displacement amplitude X at high frequency than is required at low frequency. In fact, a doubling of frequency requires a 4 times larger force to oscillate at the same displacement magnitude X.

The mid example shows how we can write the force reaction of a spring that oscillates with the frequency f. The equation reveals that the force needed to drive the spring with the oscillation amplitude X is the same at any frequency.

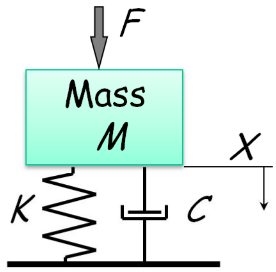
The example at the bottom shows how we can write the force reaction of a spring that oscillates with the frequency f. The equation reveals that the force needed to drive the spring with the oscillation amplitude X increases linearly with frequency.

We use the **frequency domain** conversion between displacement, velocity and acceleration because it is much simpler than the **time domain** derivatives that must otherwise be used.

A SDOF system

- Adding the mass, stiffness and damper elements together gives us:

$$\underbrace{-\omega^2 MX}_{\text{Mass force}} + \overbrace{j\omega CX}^{\text{dashpot force}} + \underbrace{KX}_{\text{spring force}} = \overbrace{F}^{\text{applied force}}$$



- The force at the ground is:

$$F_{ground} = F_{dashpot} + F_{spring} = j\omega CX + KX$$

Man lifting a police force on his shoulders

POLICE FORCE?

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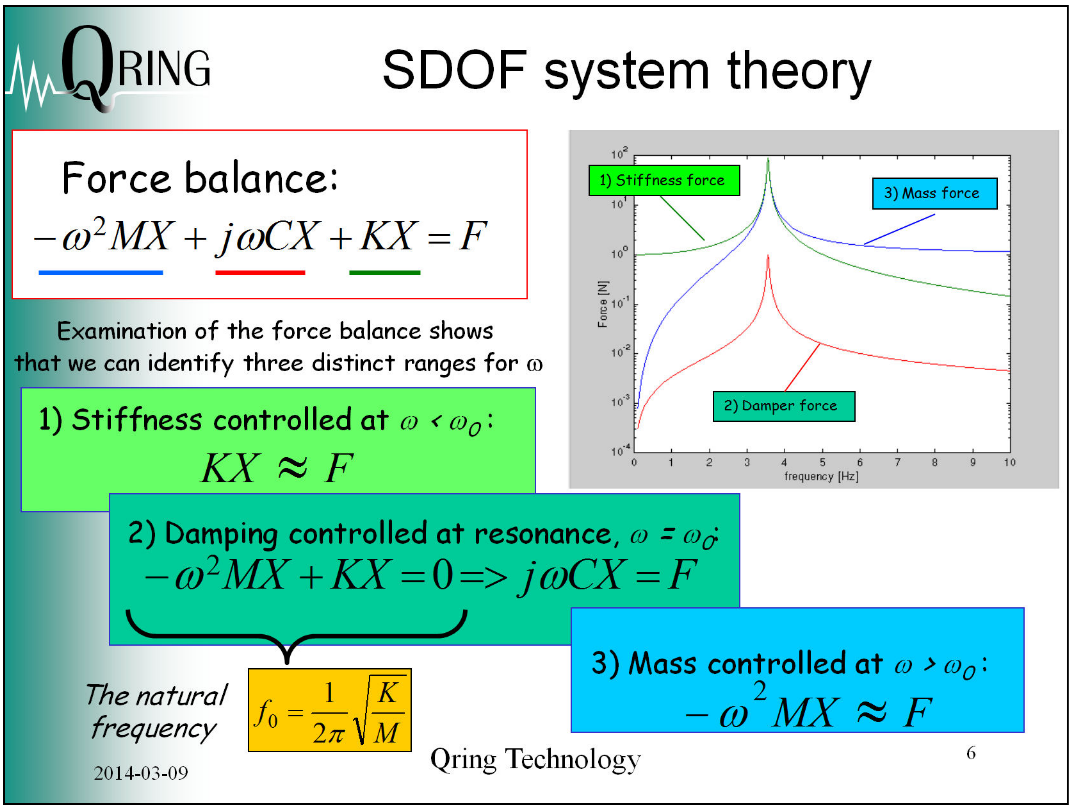
Adding the forces together gives us the fundamental force balance that is shown at the top.

The top equation tells us that the applied force is balanced by reaction from the mass, spring and damper elements. The equation tells us also that the force distribution between those varies with frequency.

The equation at the bottom shows the force between ground and the spring and the dashpot components.

This system is called a **Single Degree of Freedom** (SDOF) system because we allow it only to oscillate in a single direction (up and down), i.e. the system is free to move only in one direction.



The mass force is small when ω is small (0 Hz) but become larger as frequency is increased.

Note that the sum of the mass, spring and dashpot forces is always the same as the applied force, but that the individual forces for a component can be much larger than the applied force since the mass and spring forces have opposite signs.

The mass and spring forces are equally large and cancel perfectly at the natural frequency (also called eigen-frequency), f_0. This type of high internal dynamic loads is one reason why structures can break at resonance.

The only force left to oppose the applied force is at the natural frequency is the force of the damper. The oscillation displacement becomes large is the opposing damping force is small. A large displacement implies that the spring force, and therefore also the mass force must is large at **resonance**.

The curves in the figure shows how the forces on the individual components vary with frequency. This variation is typical for <u>any</u> SDOF system.

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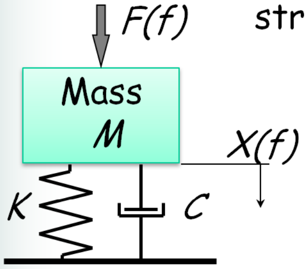
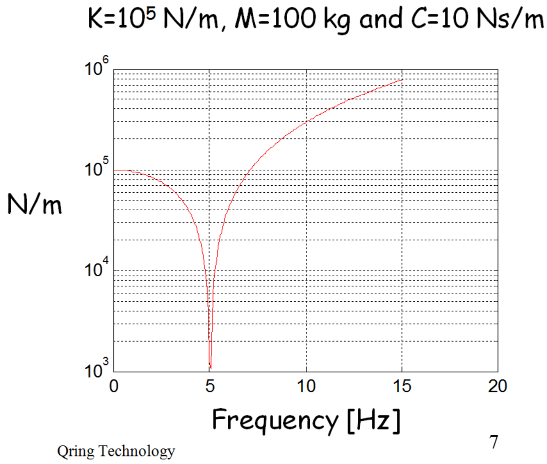
Dynamic stiffness

(Wavelength >> Dimension of the structure)





$$S = \frac{F}{X} = -\omega^2 M + j\omega C + K$$

The example shows that the dynamic stiffness of system that has weight and stiffness varies with frequency.

The fact that the dynamic stiffness can drop significantly in magnitude is one reason why dynamic problems can arise.

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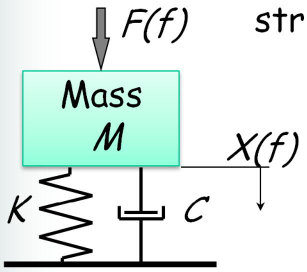
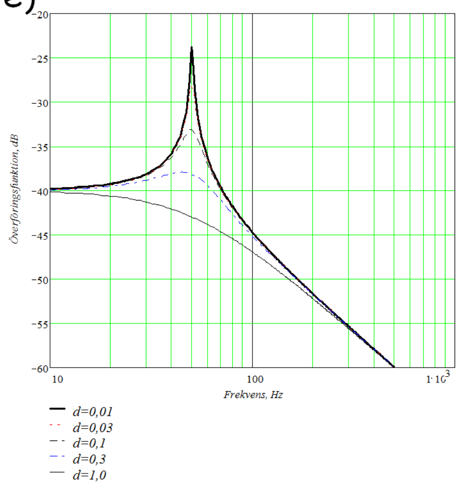
(Wavelength >> Dimension of the structure)





$$H = \frac{X}{F} = \frac{1}{-\omega^2 M + j\omega C + K}$$

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Dynamic receptance (or compliance) X/F is the inverse of dynamic stiffness F/X for the SDOF system.

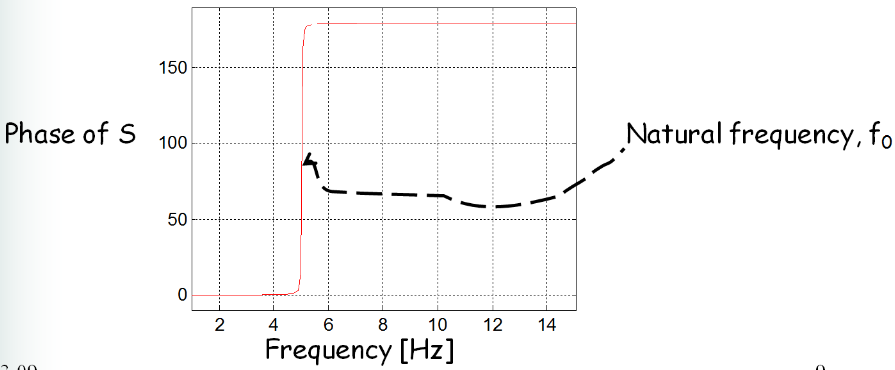
The dynamic receptance can be interpreted as the displacement we get for a 1 N force excitation at different frequencies. The FRF shows that the displacement is not the same at all frequencies.

The dynamic receptance is seen to peak at resonance. This peak can be seen to reduce as the damping C is increased.

Dynamic receptance is in the format we use when we measure dynamic response FRFs.

The phase of a SDOF system

- The figure below shows that there is a 180 degrees phase change in the FRF when we pass resonance.
- The resonance frequency is at the frequency where the phase of S (= F/X) is 90 degrees.
- This jump in phase is typical for dynamic systems and can be found in any structure.



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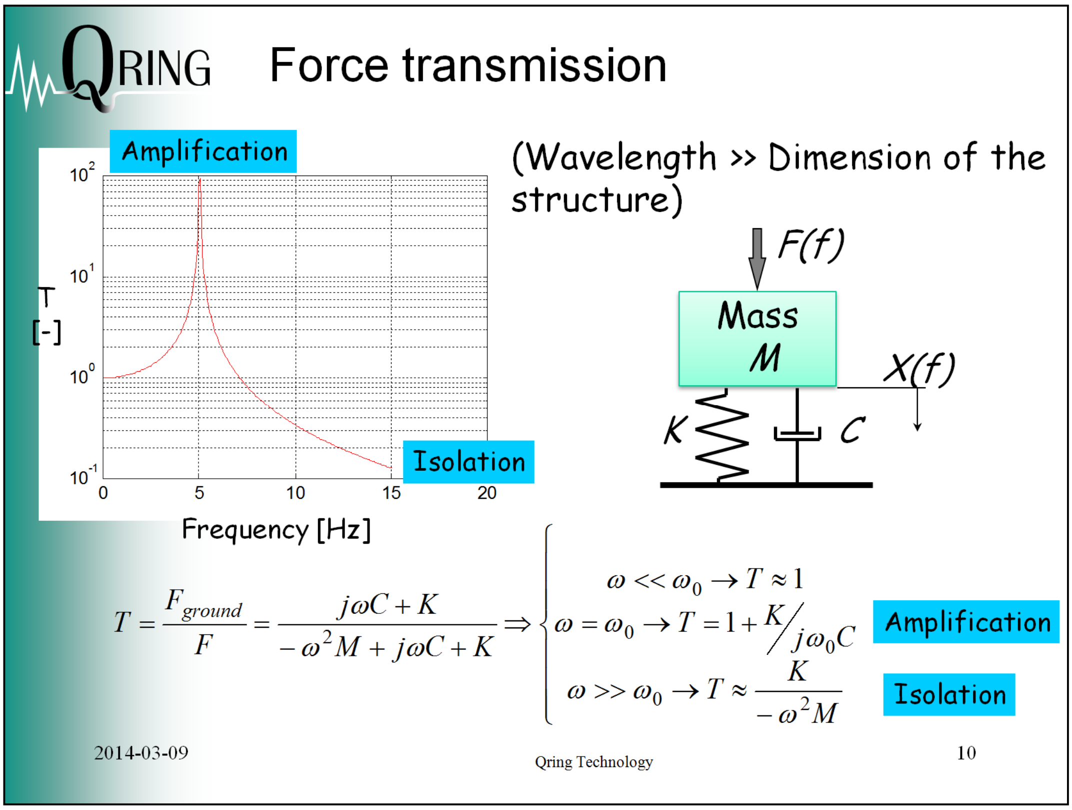
The fact the there is a 180 degree phase shift as we cross resonance and that the resonance frequency is located at the phase 90 degrees is used to identify resonance frequencies in experimental modal analysis software and FFT analysers.

Observe that the 180 degree phase shift is rather dramatic as it means that the oscillation starts to vibrate in the other direction as soon as we pass the natural frequency.



The Transmissibility equation, T shows that we have no force isolation at low frequency (T = 1).

The Transmissibility equation, T shows that the foundation must take a larger dynamic force than is applied by the force, F at the top of the SDOF system (T = 100) at resonance.

The Transmissibility equation, T shows that we receive force isolation at frequencies higher than the resonance frequency f_0 (T < 1), i.e. that the force to ground is smaller than the force that is applied on top of the SDOF system.

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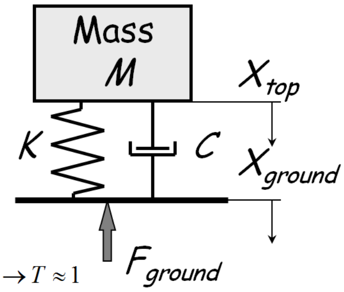
A base excited SDOF system

- Adding the mass, stiffness and damper elements together gives us:

$$-\omega^2 M X_{top} + j\omega C(X_{top} - X_{ground}) + K(X_{top} - X_{ground}) = 0$$



- Which can also be written as:

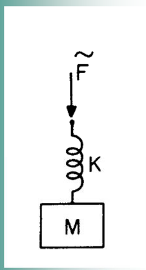
$$T = \frac{X_{top}}{X_{ground}} = \frac{j\omega C + K}{-\omega^2 M + j\omega C + K} \Rightarrow \begin{cases} \omega << \omega_0 \rightarrow T \approx 1 \\ \omega = \omega_0 \rightarrow T = 1 + K/j\omega_0 C \\ \omega >> \rightarrow T \approx \dfrac{K}{-\omega^2 M} \end{cases}$$

Note the vibration transmission expression is identical with the force transmission equation on slide **

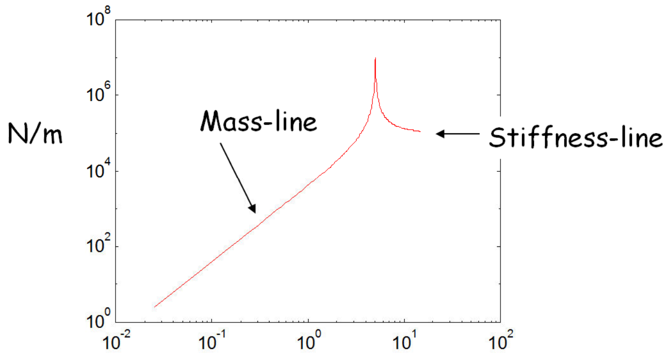
The Transmissibility equation, T shows that the vibration of the mass is larger than the vibration at ground at resonance and that we receive vibration isolation at frequencies above resonance, f_0.

Also, the Transmissibility equation, T shows that the spring must take a large dynamic force at resonance as the relative displacement is large at this frequency. This is why a building can break at earth quake.

Simple oscillator *(mass-spring system)*



Dynamic stiffness of base excited system

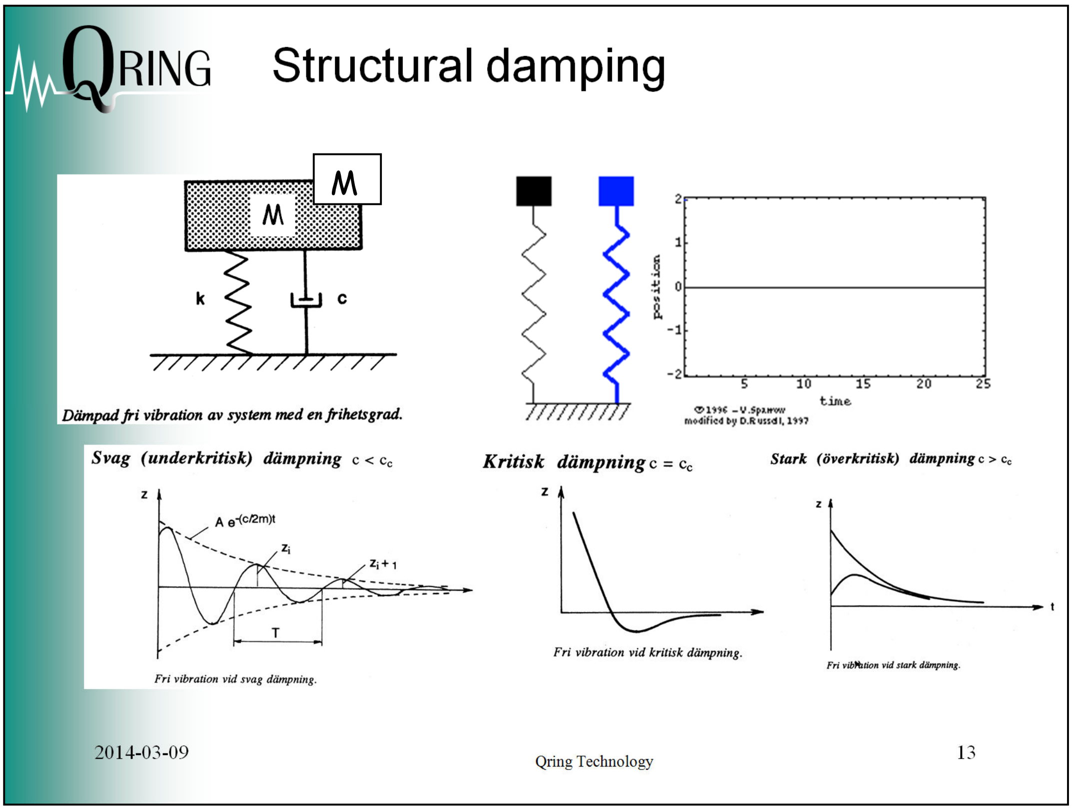


Base excited single DOF system

The force at the base senses the mass of the total system at frequencies below resonance.

The force at the base senses the spring stiffness at frequencies above resonance.

The dynamic stiffness of the base<excited system is especially high at resonance. This phenomenon of dynamic stiffening is used in tuned dampers. Tuned dampers will be discussed in a separate chapter.



Damping:

Damping is the rate at which we convert vibration energy into heat. The vibration amplitude decays as we loose vibration energy into heat.

The animation shows that the left (black) SDOF system has less damping than the SDOF system on the right since the vibration amplitude decays less rapidly when the masses are tensioned and released.

The amount of damping is measured by the vibration amplitude drop the SDOF system experiences over a time period, T.

Critical damping is the amount of damping where the mass completes one time period, T before it goes to rest. This can be seen as you lift the mass, release it, and watch the SDOF mass drop, then rise again and return to equilibrium where it rests. This is motion pattern is depicted in the critical damping figure above.

Overcritical damping means that the system is so damped that its vibration will not complete one time period, T when it is released. The motion pattern of the SDOF mass is depicted in the figure at the lowest right corner.